Poisson Power Tesselations

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Abstract: We consider generalization of the Voronoi diagram - power diagram, constructed with respect to a Poisson processes with i.i.d. marks (weights). We give first moment of the volume distribution of a typical cell, the probability that a cell is empty, the mean length of a typical edge in the planar case and other geometrical characteristics of this tesselation.

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(Résumé : tsvp)

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Tesselations poissoniennes de la puissance

Résumé : On considère le diagramme de puissance. Il s'agit d'une généralisation de la tessélation de Voronoï construite par rapport à un processus poissonien marqué par des poids i.i.d. Nous donnons les premiers moments de la distribution du volume de la cellule typique, de la probabilité qu'une cellule soit non vide, de la longueur moyenne d'une arête typique pour la tessélation plane ainsi que d'autres caractéristiques géométriques de cette tessélation.

Mots-clé : Diagrammes de puissance, tesselation de Voronoï , processus de Poisson, tesselations planes.

Introduction

Voronoi-Poisson tesselations are a classical object to study in stochastic geometry which proved to be very useful in many different fields. Their applications vary from crystallography to the Universe structure. The reader is advised to consult the book [8] to get feeling of practical importance of the subject.

Power tesselations (or diagrams) are natural generalization of Voronoi tesselations. It is really natural since they possess the basic property of the Voronoi tesselation to be an aggregate of polytopes or cells, but unlike the Voronoi ones the polytopes in a power tesselation have different weights. The bigger is the weight the larger is the corresponding cell.

Non-stochastic power diagrams were considered recently by many authors. References and basic properties can be found in [1, 2] and especially in a survey [3]. Nevertheless it seems nobody has considered yet a stochastic version which we call *Poisson power tesselations* since now diagrams are constructed with respect to a homogeneous Poisson ensemble of particles marked with independent identically distributed weights. The analysis of this class of tesselations is the object of our study here. For the sake of conveniency we give the results in Section 1 while for the proofs the reader is refered to Section 2.

1 Description of the model and results

We start with definition of Poisson power tesselations. Consider a Poisson ensemble of points - *particles* in $\mathbb{R}^d$ of intensity $\lambda > 0$. To each particle y_i of the ensemble we assign a positive number $w(y_i)$ - its *weight*. Values $w(y_i)$ are mutually independent identically distributed random variables with a distribution function F. In this case a set of particles with their weights can be interpreted as a realization Φ of a Poisson point process in a phase space $\mathcal{A} = R^d \times R_+$ driven by *the intensity measure* $\mu(da) = \lambda\ dx\ F(dw)$, where dx is Lebesgue measure in $\mathbb{R}^d$. This process is uniquely defined by probabilities

$$\mathbf{P}\{\Phi(X \times W) = n\} = \frac{(\lambda|X|\mathbf{P}_0(W))^n}{n!}\exp(-\lambda|X|\mathbf{P}_0(W)),\ n = 0, 1, 2\dots$$

for all bounded Borel $X \subset \mathbb{R}^d$, $W \subset \mathbb{R}$, where $\Phi(X \times W)$ is the number of particles y_i of Φ such that $(y_i, w(y_i)) \in X \times W$ and $|X|$ stands for Lebesgue measure of a set X.

Another useful interpretation of Φ is that of *a Poisson marked process* with independent marks w. A *mark* $m(x_i)$ of a particle x_i is some characteristic of the process Φ related to x_i. It can be its weight, geometrical characteristics of the corresponding cell etc. Since the underlying Poisson process of particles is stationary, then one can define a

Palm distribution $\mathbf{P}_0$ of marks, which is roughly speaking, the conditional distribution of $m(o)$ given a particle at 0 (cf. [9, pp.101-104]). In ergodic case the Palm distribution can be interpreted as the distribution of a mark of a "typical particle" (cf. [4, p.339]). When considering the weights as marks of particles and using their independence one can conclude that their Palm distribution coincides with the distribution $F(dw)$. That is why two forms writing $\int f(x)F(dx)$ and $\mathbf{E}_0 f(w)$ are equivalent, where $\mathbf{E}_0$ is the expectation with respect to the Palm measure.

For any point $x \in \mathbb{R}^d$ and a particle y define a function $pow\,(x, y, w(y)) = pow\,(x, a) \stackrel{\text{def}}{=} |x-y|^2 - w(y)$ - *a power* of a point x with respect to a pair $a = (y, w(y))$. The used term corresponds to the well known geometrical notion of a power of a point x with respect to a sphere $S(y)$ centered in y of radius $\sqrt{w(y)}$. Time to time we will use this geometrical interpretation of a power. Given two particles y_i and y_j *a chordale* of y_i and y_j is a set $chor\,(y_i, y_j) = \{x \in \mathbb{R}^d \mid pow\,(x, y_i, w(y_i)) = pow\,(x, y_j, w(y_j))\}$ which is actually a hyperplane perpendicular to the line through the points y_i, y_j. Moreover, if $S(y_i) \cap S(y_j) \neq \emptyset$, then $S(y_i) \cap S(y_j) \subset chor\,(y_i, y_j)$ (for details see [1]).

A cell with *nucleus* y_0 of weight $w_0 = w(y_0)$ is the set

$$C(y_0) = \{x \in \mathbb{R}^d \mid pow\,(x, y_0, w_0) \leq pow\,(x, y_i, w(y_i)) \text{ for all particles } y_i\}.$$

Being an intersection of half-spaces, $C(y_0)$ is a convex set, but possibly empty. Moreover, unlike the Voronoi tesselations, $C(y_0)$ may not include its nucleus y_0.

We say that a countable ensemble of polytops $\{C_i\}$ constitutes *a tesselation* of space, if $\mathbb{R}^d = \cup_{i=1}^{\infty} C_i$ and d-content of the boundaries of C_i is zero. Tesselation is *simple* if any of its j-dimensional face lies in the closure of exactly $d - j + 1$ polytops.

A typical configuration of a Poisson power tesselation in non-degenerate case given by conditions of Statement 1 below is shown on Figure 1.

The following statement gives a consistency condition of the model under consideration.

Statement 1 *The marked Poisson process Φ defines a power tesselation of $\mathbb{R}^d$ if and only if $\mathbf{E}_0 w^{d/2} < \infty$. This tesselation is almost surely simple when exists.*

Statement 2 *The mean volume of a typical cell is given by*

$$\mathbf{E}_0 vol\,(C) = s_{d-1} \int_0^{\infty} r^{d-1} \mathbf{E}_0\, \psi_\lambda(r^2 - w_0)\, dr, \tag{1}$$

$$\text{where } \psi_\lambda(\beta) = \exp\left\{-\lambda v_d \mathbf{E}_0\{[(w+\beta)^+]^{d/2}\}\right\}\, dr \tag{2}$$

and w_0, w are i.i.d. random variables with distribution F.

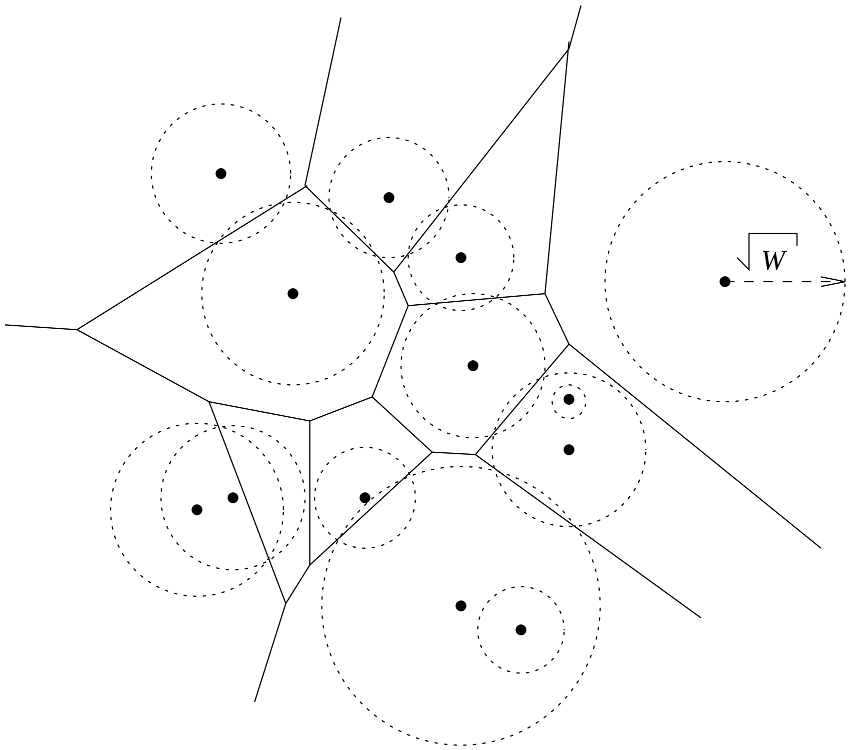

Figure 1

Let λ_0 denote the density of cells. Then the standard Palm theory considerations (see [7]) give $\lambda_0^{-1} = \mathbf{E}_0 vol\,(C) = \lambda p_0$, where p_0 is the Palm probability that the cell of a particle 0 is nonempty. Therefore $p_0 = (\lambda \mathbf{E}_0 vol\,(C))^{-1}$. Since λ is just a scale parameter then increasing λ A times is equivalent to simultaneous increase of all the powers A times, that does not change the geometry of the tesselation. Thus p_0 must not depend on λ and we come to the following

Statement 3 *Palm probability that the cell is nonempty is given by*

$$p_0 = \left[s_{d-1} \int_0^\infty r^{d-1} \mathbf{E}_0 \psi_1(r^2 - w_0)\, dr \right]^{-1} \tag{3}$$

where the function ψ_1 is defined by (2)*.*

Now we fix upon a planar case $d = 2$ for which all the mean value relationships could in principal be evaluated explicitly.

Statement 4 *For the power tesselation in a plane* $\mathbb{R}^2$

(i) the mean length of a tesselation edge between particles of powers w_1 *and* w_2 *is given by*

$$\bar{l_1}(w_1, w_2) = 2\pi \int_0^\infty \rho\, d\rho \int_0^\infty \psi_\lambda(h_{w_1,w_2}(z,\rho))\, dz\,; \tag{4}$$

(ii) the mean length of a tesselation edge equals

$$\bar{l_1} = \mathbf{E}_0^2\, \bar{l_1}(w_1, w_2)\,; \tag{5}$$

(iii) the mean length of all edges emanating from a vertex is equal to

$$\bar{l_0} = 3\mathbf{E}_0^2\, \bar{l}(w_1, w_2)\,; \tag{6}$$

(iv) the mean perimeter of a cell is equal to

$$\bar{l_2} = 6\mathbf{E}_0^2\, \bar{l}(w_1, w_2)\,; \tag{7}$$

(v) the mean length of edges per unit area equals

$$L_A = 3\lambda p_0 \bar{l_1}\,; \tag{8}$$

(vi) intensities of vertices, edges and cells are given respectively by

$$\lambda_0 = \lambda p_0 \tag{9}$$
$$\lambda_1 = 3\lambda p_0 \tag{10}$$
$$\lambda_2 = 2\lambda p_0 \tag{11}$$

where

$$h_{w_1,w_2}(z,\rho) = \frac{(w_1 - w_2 + \rho^2)^2}{4\rho^2} - w_1 + z^2,$$
$$\mathbf{E}_0^2\, g(w_1, w_2) = \int\int g(w_1, w_2)\, F(dw_1)\, F(dw_2),$$

and ψ, p_0 *are defined by* (2) *and* (3) *respectively.*

In principal, there is no problem to write expressions for higher order moments, but even in the case of Voronoi tesselations they involve numerical integration (see [6]). For general models we consider they become really cumbersome.

2 Proofs

Proof of Statement 1. First of all note that if the tesselation exists it is simple with probability 1. This is an easy consequence of the fact, that particles in a Poisson ensemble are almost surely in general position.

Fix now a point $x \in \mathbb{R}^d$ and a constant β. Using independence of weights for different particles we may write

$$\begin{aligned} & \mathbf{P}\{pow\,(x,a_i) > \beta \text{ for all } a_i = (y_i, w(y_i)) \in \Phi\} \\ = \; & \mathbf{E} \prod_{a_i \in \Phi} \mathbf{1}\{pow\,(x,a_i) > \beta\} = \exp\left\{-\int_{\mathcal{A}} \mathbf{1}\{pow\,(x,a) > \beta\}\,\mu(da)\right\} \end{aligned}$$

To get the last equality we have used the explicit expression for the generating functional $G[h] \stackrel{\text{def}}{=} \mathbf{E}\prod_{a_i\in\Phi} h(a_i)$ of a point process Φ which for the Poisson case is known to be equal to

$$G[h] = \exp\left\{-\int_{\mathcal{A}} (1-h(a))\,\mu(da)\right\}$$

(cf. [4, p. 225]).

Switching to the polar coordinates $r = |x-y|$ and using notations v_d for the volume of the unit d-dimensional ball and $s_{d-1} = dv_d$ for the surface content of the (d-1)-dimensional unit sphere, we find that

$$\begin{aligned} & \int_{\mathcal{A}} \mathbf{1}\{|x-y|^2 - w < \beta\}\,dy\,F(dw) \\ = \; & \lambda s_{d-1} \int_0^\infty r^{d-1} \mathbf{P}_0\{w > r^2 - \beta\}\,dr = \lambda v_d \int_0^\infty \mathbf{P}_0\{w > t-\beta\}\,dt^{d/2} \\ = \; & \lambda v_d \lim_{t\to\infty} (1 - F(t-\beta))t^{d/2} + \lambda v_d \int_0^\infty t^{d/2} dF(t-\beta) \end{aligned}$$

which is infinite for all β when $\mathbf{E}_0 w^{d/2} = \infty$. In this case

$$\min_{y_i} pow\,(x, y_i, w(y_i)) = -\infty$$

for all $x \in \mathbb{R}^d$ with probability 1 and hence a cell $C(y_i)$ is empty for any particle $y_i \in \Phi$.

If $\mathbf{E}_0 w^{d/2} < \infty$ then the last expression equals

$$\lambda v_d \int_{-\infty}^\infty [t^+]^{d/2} dF_{w+\beta}(t) = \lambda v_d \mathbf{E}_0[(w+\beta)^+]^{d/2},$$

where $t^+ \stackrel{\text{def}}{=} \max\{t, 0\}$. Thus we have found that

$$\begin{aligned} & \mathbf{P}\{pow\,(x, a_i) > \beta \text{ for all } a_i = (y_i, w(y_i)) \in \Phi\} \\ = \; & \exp\left\{-\lambda v_d \mathbf{E}_0[(w+\beta)^+]^{d/2}\right\} \stackrel{\text{def}}{=} \psi_\lambda(\beta) \end{aligned} \qquad (12)$$

Since

$$\mathbf{P}\{\min_{y_i} pow\,(x, y_i, w(y_i)) = -\infty\} = \lim_{\beta \to \infty} (1 - \psi_\lambda(\beta)) = 0$$

then for any x there defined at least one y_i for which $pow(x, y_i, w(y_i))$ is minimal. Thus x is contained in $C(y_i)$. Moreover, since in a homogeneous Poisson process the particles are in general position, then $C(y_i) \cap C(y_j)$ for all $i \neq j$ has Lebesgue d-measure 0 and thus the cells form a tesselation of $\mathbb{R}^d$ (see [1]).

□

Proof of Statement 2. A point x at the distance r of the origin belongs to a cell of the particle 0 if

$$pow\,(x, a_i) \geq pow\,(x, 0, w_0) = r^2 - w_0 \text{ for all } a_i = (y_i, w(y_i)) \in \Phi$$

Probability of this event given the power w_0 of 0 is $\psi_\lambda(r^2 - w_0)$ where $\psi_\lambda(\beta)$ is defined by (2). Now (1) is immediate after integration over all possible w_0 and positions of x.

□

Remark 1 The statement of Statement 1 matches well with the fact known in the percolation theory. The value of $\mathbf{E}_0 w^{d/2}$ coincides up to a positive factor with expectation of the volume of a ball of radius $\sqrt{w}$. It is known that in the case when this expectation is infinite, each point is hitted by infinitely many of these balls centered in particles of the Poisson ensemble [5]. Since the power function is negative inside the corresponding ball then one may expect

$$\min_{y_i} pow\,(x, y_i, w(y_i)) = -\infty$$

for all x, that is really the case.

Proof of Statement 4. Consider a pair of points $P_i,\ i = 1, 2$ in $\mathbb{R}^2$. Let $A(a_1, a_2)$ be the center of segment P_1P_2, $\rho \in [0, \infty)$ be its length and $\phi \in [0, \pi)$ denote its polar angle i. e. the angle between the segment P_1P_2 and the axis Ox. Then $dP_1 dP_2 = dA\, \rho d\rho\, d\phi$. Suppose now that P_1, P_2 describe position of two particles with weights w_1 and w_2 respectively.

Taking the point A as a mark of an edge separating the cells $C(P_1)$ and $C(P_2)$ we may write that

$$\bar{l}(w_1, w_2) = \int\int\int \mathbf{P}\{P_0 \in chor\,(P_1, w_1, P_2, w_2) \cap C(P_1) \cap C(P_2) \mid A \in 0\} dP_0\, \rho d\rho\, d\phi$$

Parametrize the position of the point P_0 by the distance z to the point $(P_1P_2) \cap chor\,(P_1, w_1, P_2, w_2)$ (see fig.2) It is easy to check that

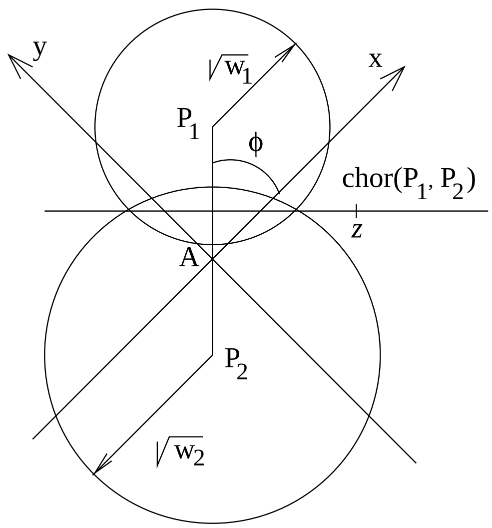

Figure 2

$$pow\,(P_0, P_1, w_1) = pow\,(P_0, P_2, w_2)$$
$$= \frac{(w_1 - w_2 + l^2)^2}{4l^2} - w_1 + z^2 \overset{\text{def}}{=} h_{w_1,w_2}(z, \rho)$$

Using independence of position and power of particles in a Poisson process to find the latter probability it suffices now to apply formula (2) with $\beta = h_{w_1,w_2}(z, \rho)$ that proves (4) after integration with respect to ϕ.

(5) is an immediate consequence of (4). All the other statements are just the general relationships between the mean values for stationary planar tesselations (see the above cited [7]).

□

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